

SUPPRESSION OF UNBOUNDED GRADIENTS IN A SDE ASSOCIATED WITH THE BURGERS EQUATION

SERGIO ALBEVERIO ¹, OLGA ROZANOVA ²

ABSTRACT. We consider the Langevin equation describing a non-viscous Burgers fluid stochastically perturbed by uniform noise. We introduce a deterministic function that corresponds to the mean of the velocity when we keep fixed the value of the position. We study interrelations between this function and the solution of the non-perturbed Burgers equation. Especially we are interested in the property of the solution of the latter equation to develop unbounded gradients within a finite time. We study the question how the initial distribution of particles for the Langevin equation influences this blowup phenomenon. We show that for a wide class of initial data and initial distributions of particles the unbounded gradients are eliminated. The case of a linear initial velocity is particular. We show that if the initial distribution of particles is uniform, then the mean of the velocity for a given position coincides with the solution of the Burgers equation and, in particular, it does not depend on the constant variance of the stochastic perturbation. Further, for a one space variable we get the following result: if the decay rate of the even power-behaved initial particles distribution at infinity is greater or equal $|x|^{-2}$, then the blowup is suppressed, otherwise, the blowup takes place at the same moment of time as in the case of the non-perturbed Burgers equation.

1. INTRODUCTION

It is well known that the non-viscous Burgers equation, the simplest equation that models the nonlinear phenomena in a force free mass transfer,

$$u_t + (u, \nabla) u = 0, \quad (1.1)$$

where $u(x, t) = (u_1, \dots, u_n)$ is a vector-function $\mathbb{R}^{n+1} \rightarrow \mathbb{R}^n$, before the formation of shocks is equivalent to the system of ODE

$$\dot{x}(t) = u(t, x(t)), \quad \dot{u}(t, x(t)) = 0. \quad (1.2)$$

The latter system defines a family of characteristic lines $x = x(t)$ that can be interpreted as the Lagrangian coordinates of the particles.

Given initial data

$$u(x, 0) = u_0(x), \quad (1.3)$$

one can readily get an implicit solution of (1.1), (1.3), namely,

$$u(t, x) = u_0(x - tu(t, x)).$$

Date: April 27, 2009.

1991 Mathematics Subject Classification. 35R60.

Key words and phrases. Burgers equation, gradient catastrophe.

Supported by Award DFG 436 RUS 113/823/0-1 and the special program of the Ministry of Education of the Russian Federation "The development of scientific potential of the Higher School", project 2.1.1/1399.

For special classes of initial data we can obtain an explicit solution. The simplest case is

$$u_0(x) = \alpha x, \quad \alpha \in \mathbb{R}, \quad (1.4)$$

where

$$u(t, x) = \frac{\alpha x}{1 + \alpha t}. \quad (1.5)$$

Thus, if $\alpha < 0$, the solution develops a singularity at the origin as $t \rightarrow T$, $0 < T < \infty$, where

$$T = -\frac{1}{\alpha}. \quad (1.6)$$

In the present paper we consider a $2 \times n$ dimensional Itô stochastic differential system of equations, associated with (1.2), namely

$$\begin{aligned} dX_k(t) &= U_k(t) dt, \\ dU_k(t) &= \sigma d(W_k)_t, \quad k = 1, \dots, n, \\ X(0) &= x, \quad U(0) = u, \quad t \geq 0, \end{aligned}$$

where $(X(t), U(t))$ runs the phase space $\mathbb{R}^n \times \mathbb{R}^n$, $\sigma > 0$ is constant, $(W)_{k,t}$, $k = 1, \dots, n$, is the n - dimensional Brownian motion.

Our main question is: can a stochastic perturbation suppress the appearance of unbounded gradients?

The stochastically perturbed Burgers equation and the relative Langevin equation were treated in many works (e.g. [1],[2]). The behavior of the space gradient of the velocity was studied earlier in other contexts in [3], [4], but this problem is quite different from the problem considered in the present paper. The analogous problem concerning the behavior of gradients of solutions to the Burgers equation under other types of stochastic perturbations was studied in [5].

Let us consider the mean of the velocity $U(t)$ at time t when we keep the value of $X(t)$ at time t fixed but allow $U(t)$ to take any value it wants, namely

$$\hat{u}(t, x) = \frac{\int_{\mathbb{R}^n} u P(t, x, u) du}{\int_{\mathbb{R}^n} P(t, x, u) du}, \quad t \geq 0, x \in \mathbb{R}^n, \quad (1.7)$$

where $P(t, x, u)$ is the probability density in position and velocity space, so that $\int_{\mathbb{R}^n \times \mathbb{R}^n} P(t, x, u) dx du = 1$.

This function obeys the following Fokker-Planck equation:

$$\frac{\partial P(t, x, u)}{\partial t} = \left[-\sum_{k=1}^n u_k \frac{\partial}{\partial x_k} + \frac{1}{2} \sigma^2 \frac{\partial^2}{\partial u_k^2} \right] P(t, x, u), \quad (1.8)$$

subject to the initial data

$$P(0, x, u) = P_0(x, u).$$

If we choose

$$P_0(x, u) = \delta(u - u_0(x)) f(x) = \prod_{k=1}^n \delta(u_k - (u_0(x))_k) f(x), \quad (1.9)$$

with an arbitrary sufficiently regular $f(x)$, then $\hat{u}(0, x) = u_0(x)$. The function $f(x)$ has the meaning of a probability density of the particle positions in the space at the initial moment of time and therefore $f(x)$ has to be chosen such

that $\int_{\mathbb{R}^n} f(x) dx = 1$. If the latter integral diverges for a certain choice of $f(x)$, we consider the domain $\Omega_L := [-L, L]^n$, $L > 0$ and the re-normalized density $f_L(x) := \chi(\Omega_L) f(x) \left(\int_{\Omega_L} f(x) dx \right)^{-1}$, where $\chi(\Omega_L)$ is the characteristic function of Ω_L , we denote the respective probability density in velocity and position by $P_L(t, x, u)$ and modify the definition of $\hat{u}(t, x)$ as follows:

$$\hat{u}(t, x) = \lim_{L \rightarrow \infty} \frac{\int_{\mathbb{R}^n} u P_L(t, x, u) du}{\int_{\mathbb{R}^n} P_L(t, x, u) du}, \quad t \geq 0, x \in \Omega_L, \quad (1.10)$$

provided the limit exists.

We apply heuristically the Fourier transform in the variables u and x to (1.8), (1.9) to obtain for $\tilde{P} = \tilde{P}(t, \lambda, \xi)$

$$\frac{\partial \tilde{P}}{\partial t} = -\frac{\sigma^2}{2} |\xi|^2 \tilde{P} + (\lambda, \frac{\partial \tilde{P}}{\partial \xi}), \quad (1.11)$$

$$\tilde{P}(0, \lambda, \xi) = \int_{\mathbb{R}^n} f(s) e^{-i(\xi, u_0(s))} e^{-i(\lambda, s)} ds. \quad (1.12)$$

Thus, (1.11) and (1.12) give

$$\tilde{P}(t, \lambda, \xi) = e^{-\frac{\sigma^2}{6|\lambda|}(|\xi + \lambda t|^3 - |\xi|^3)} \int_{\mathbb{R}^n} f(s) e^{-i((\xi + \lambda t), u_0(s))} e^{-i(\lambda, s)} ds, \quad (1.13)$$

$$\begin{aligned} P(t, x, u) &= \frac{1}{(2\pi)^{2n}} \int_{\mathbb{R}^{2n}} \tilde{P}(t, \lambda, \xi) e^{i(\xi, u)} e^{i(\lambda, x)} d\lambda d\xi = \\ &= \left(\frac{\sqrt{3}}{\pi \sigma^2 t^2} \right)^n \int_{\mathbb{R}^n} f(s) e^{-\frac{3}{\sigma^2 t^3} (3t^2 (u, u_0(s)) + t^2 |u_0(s) - u|^2 + 3|x - s|^2 + 3t (u + u_0(s), s - x))} ds. \end{aligned} \quad (1.14)$$

Now we substitute (1.14) in (1.8) or (1.10), integrate with respect to u and get the formula

$$\hat{u}(t, x) = \frac{1}{2t} \frac{\int_{\mathbb{R}^n} (-u_0(s)t - 3(s - x)) f(s) e^{-\frac{3|u_0(s)t + (s-x)|^2}{2\sigma^2 t^3}} ds}{\int_{\mathbb{R}^n} f(s) e^{-\frac{3|u_0(s)t + (s-x)|^2}{2\sigma^2 t^3}} ds}, \quad t \geq 0, x \in \mathbb{R}^n, \quad (1.15)$$

provided all integrals exist.

Thus, we can compare $\hat{u}(t, x)$ with the solution $u(t, x)$ of the non-viscous Burgers equation (1.1).

2. EXACT RESULTS

It is natural to begin with the case where the solution to the Burgers equation (1.1) can be obtained explicitly. Let us choose

$$u_0(x) = \alpha x, \quad \alpha < 0. \quad (2.1)$$

One can see from (1.5), (1.6) that the gradient of the solution becomes unbounded as $t \rightarrow T$.

If the initial distribution of particles is either uniform or Gaussian, it is possible to get explicit formulas for \hat{u} . Namely, for the uniform distribution $f(x) = \text{const}$ both integrals in the numerator and the denominator in (1.15) can be taken and we get

$$\hat{u}(t, x) = \frac{\alpha x}{1 + \alpha t},$$

which coincides with (1.5). Therefore, the gradient becomes unbounded at $T = -\frac{1}{\alpha}$. On the contrary, in the case of a Gaussian distribution, $f(x) = \left(\frac{r}{\sqrt{\pi}}\right)^n e^{-r^2|x|^2}$, $r > 0$, we get another explicit formula:

$$\hat{u}(t, x) = \frac{3(\alpha(\alpha t + 1) + r^2\sigma^2 t^2)}{3(\alpha t + 1)^2 + 2r^2\sigma^2 t^3} x. \quad (2.2)$$

One can see that the denominator does not vanish for all positive t , and at the critical time T we have $\hat{u}(t, x) = -\frac{3}{2}\alpha x$, that is the gradient becomes positive and tends to zero as $t \rightarrow +\infty$.

3. 1D CASE, SPECIFIC CLASSES OF INITIAL DISTRIBUTIONS OF PARTICLES AND INITIAL DATA

Our main question is how the decay rate of the function $f(x)$ at infinity relates to the property of \hat{u} to reproduce the behavior of the solution of the non-perturbed Burgers equation at the critical time. For the sake of simplicity we dwell on the case of an one dimensional space, however the results can be extended to the higher dimensional space. Let us consider the class of initial distributions of particles $f(x)$ which are intermediate between Gaussian and uniform. Our aim is to find a threshold rate of decay at infinity that still allows to preserve the singularity at the origin.

We restrict ourselves to the class of smooth distributions $f(x)$ and initial data $u_0(x)$ satisfying the condition

$$\left| \int_{\mathbb{R}} \xi^m (u_0(\xi))^l f(\xi) \exp(-\gamma \xi^2) d\xi \right| < \infty \quad \text{for all } m, l \in \mathbb{N} \cup \{0\}, \gamma > 0. \quad (3.1)$$

As a representative of such a class of distributions we can consider

$$f(x) = \text{const} \cdot (1 + |x|^2)^k, \quad k \in \mathbb{R}. \quad (3.2)$$

Theorem 3.1. *Let the initial be $u_0(x)$ be smooth, and for a certain fixed $\beta < 0$ and all $x \in \mathbb{R}$ (except for maybe a bounded set) $|u_0(x) - \beta x| \geq \gamma > 0$. Moreover, assume that the distribution function $f(x)$ is smooth, nonnegative, and the property (3.1) is satisfied. Then the mean $\hat{u}(t, x)$ has at the origin $x = 0$ at the moment $t_0 = -\frac{1}{\beta}$, $\beta < 0$, a bounded derivative $u'_x(t_0, 0)$.*

We remark that the initial data with a linear initial profile except for $\beta = u'_0(x)$ fall into the class of initial data that we have described above.

Proof. First of all we perform a change of the time variable. Let $\epsilon = t + \frac{1}{\beta}$, $\beta < 0$. We expand $\hat{u}(t(\epsilon), x)$ given by (1.15) into Taylor series at the point $t = t_0 = -\frac{1}{\beta}$ ($\epsilon =$

0), $x = 0$, taking into account that condition (3.1) guarantees the convergence of the integrals in the coefficients of the expansion. This expansion has the form

$$\begin{aligned} \hat{u}(t(\epsilon), x) \sim & \frac{1}{2} \frac{\int_{\mathbb{R}} (3s\beta + u_0(s)) f(s) e^{\frac{3\beta^3}{2\sigma^2} \left(\frac{u_0(s)}{\beta} - s\right)^2} ds}{\int_{\mathbb{R}} f(s) e^{\frac{3\beta^3}{2\sigma^2} \left(\frac{u_0(s)}{\beta} - s\right)^2} ds} + \\ & - \frac{3\beta}{2\sigma^2} \left(\frac{\int_{\mathbb{R}} (\sigma^2 - 4\beta^2 s u_0(s) + 3s^2 \beta^3 + \beta(u_0(s))^2) f(s) e^{\frac{3\beta^3}{2\sigma^2} \left(\frac{u_0(s)}{\beta} - s\right)^2} ds}{\int_{\mathbb{R}} f(s) e^{\frac{3\beta^3}{2\sigma^2} \left(\frac{u_0(s)}{\beta} - s\right)^2} ds} + \right. \\ & \left. \frac{\int_{\mathbb{R}} (\beta s - u_0(s)) f(s) e^{\frac{3\beta^3}{2\sigma^2} \left(\frac{u_0(s)}{\beta} - s\right)^2} ds}{\int_{\mathbb{R}} f(s) e^{\frac{3\beta^3}{2\sigma^2} \left(\frac{u_0(s)}{\beta} - s\right)^2} ds} \right) x, \end{aligned} \quad (3.3)$$

as $x \rightarrow 0$, $\epsilon \rightarrow 0-$ (where \sim stands for the quotient of the left and right sides converging to 1).

The theorem follows immediately from the asymptotics (3.3). \square

Let us notice that for even $f(x)$ and odd $u_0(x)$ the expansion (3.3) is less cumbersome, namely,

$$\hat{u}(t(\epsilon), x) \sim - \frac{3\beta}{2\sigma^2} \frac{\int_0^\infty (\sigma^2 - 4\beta^2 s u_0(s) + 3s^2 \beta^3 + \beta(u_0(s))^2) f(s) e^{\frac{3\beta^3}{2\sigma^2} \left(\frac{u_0(s)}{\beta} - s\right)^2} ds}{\int_0^\infty f(s) e^{\frac{3\beta^3}{2\sigma^2} \left(\frac{u_0(s)}{\beta} - s\right)^2} ds} x, \quad (3.3^*)$$

as $x \rightarrow 0$, $\epsilon \rightarrow 0-$.

It can be readily calculated that if $\beta \rightarrow -\infty$ ($t_0 \rightarrow 0$), then (3.3*) yields $\hat{u}(t, x) \sim \alpha x$, $x \rightarrow 0$, $\epsilon \rightarrow 0-$, where $\alpha = u_x(0)$ (taking account of $\frac{u_0(\xi)}{\xi} \sim \alpha$, $u'_0(\xi) \sim \alpha$, $\xi \rightarrow 0$).

3.1. Power-behaved distribution. Let us consider the specific class of even distributions (3.2) and linear initial data.

The case a linear initial function $u_0(x) = \alpha x$, $\alpha \neq \beta$, is particular. Indeed, we have from (3.3*) for $x \rightarrow 0$ and for $t \rightarrow t_0 = -\frac{1}{\beta}$, $\beta < 0$, the following asymptotic behavior:

$$\hat{u}(t, x) \sim \Lambda(\beta) x,$$

with

$$\Lambda(\beta) = - \frac{3\beta}{2\sigma^2} \frac{\int_0^\infty (\sigma^2 + \beta s^2 (\alpha - \beta)(\alpha - 3\beta)) f(s) e^{\frac{3s^2 \beta}{2\sigma^2} (\beta - \alpha)^2} ds}{\int_0^\infty f(s) e^{\frac{3s^2 \beta}{2\sigma^2} (\beta - \alpha)^2} ds}.$$

We can see that if $\beta < \alpha$ (before the critical time $T = -\frac{1}{\alpha}$, when the solution of the non-perturbed Burgers equation blows up) or $\beta > \alpha$ (after the time T) both integrals in (3.3*) converge and therefore the derivative $\hat{u}'_x(t, 0)$ remains bounded.

Let us consider now the critical moment of time $t = T$, where $\beta = \alpha$. In this case $\frac{u_0(x)}{x} = \beta$ identically and we do not have a multiplier that guarantees the convergence of integrals of the form

$$\int_{\mathbb{R}_+} \xi^m f(\xi) d\xi \quad \text{for all } m \in \mathbb{N},$$

which is necessary for the validity of the asymptotics (3.3*).

However, fortunately, due to the relative simplicity of $f(x)$ we can compute $\hat{u}(t, x)$ in the vicinity of the origin directly, using the formula (1.15), which in this case takes the form

$$\hat{u}(t, x) = \frac{1}{2t} \frac{\int_{\mathbb{R}^+} (-\alpha st - 3(s-x)) (1+s^2)^k e^{-3|\alpha st + (s-x)|^2} ds}{\int_{\mathbb{R}^+} (1+s^2)^k e^{-3|\alpha st + (s-x)|^2} ds}, \quad t \geq 0, x \in \mathbb{R}. \quad (3.4)$$

Computations show that for $k \neq \frac{m}{2}$, $m \in \mathbb{Z}$, the asymptotic behavior of (3.4) as $x \rightarrow 0$, $\epsilon \rightarrow 0-$, where $\epsilon = t + \frac{1}{\alpha}$, can be expressed through the Gamma function and the generalized Laguerre functions $L(\nu_1, \nu_2, \nu_3)$, see [6]. It has the form

$$\hat{u}(t, x) \sim \frac{F_1(\epsilon, k, \alpha, \sigma)}{F_2(\epsilon, k, \alpha, \sigma)} x, \quad (3.5)$$

$$F_1(\epsilon, k, \alpha, \sigma) = A_1(k) \epsilon^{-2k-2} + o(\epsilon^{-2k-2}) + A_2(k) \epsilon^0 + o(\epsilon^0),$$

$$F_2(\epsilon, k, \alpha, \sigma) = A_3(k) \epsilon^{-2k-1} + o(\epsilon^{-2k-1}) + A_4(k) \epsilon^0 + o(\epsilon^0),$$

where the coefficients $A_i(k)$, $i = 1, \dots, 4$, are as follows:

$$A_1(k) = \frac{\pi^2 2^{k+2} \sigma^{2k} (4k^2 - 1)}{3^k |\alpha|^{5k+1} \cos \pi k} \Gamma(k+1) L(k, -k + \frac{1}{2}, 0),$$

$$A_2(k) = \frac{3\sqrt{6}|\pi\alpha|^{\frac{5}{2}}}{2\sigma(k+1)} \tan(\pi k) L(\frac{1}{2}, k + \frac{1}{2}, 0),$$

$$A_3(k) = \frac{\pi^2 2^{k+1} \sigma^{2k} (2k-1)}{3^k |\alpha|^{5k+1} (k+1)(k+2) \cos \pi k} \Gamma(k+3) L(k, -k + \frac{1}{2}, 0),$$

$$A_4(k) = \frac{\sqrt{6}\pi^2 |\alpha|^{\frac{3}{2}} (2k+3) \Gamma(k+3)}{\sigma(k+1)(k+2) \Gamma(k+\frac{5}{2})} \tan(\pi k).$$

Thus, if $k < -1$, then the leading term of the numerator and denominator in (3.5) as $\epsilon \rightarrow 0-$ is $A_2 \epsilon^0$ and (3.5) can be written as

$$\hat{u}(t, x) \sim \frac{A_2(k) \epsilon^0 + o(\epsilon^0)}{A_4(k) \epsilon^0 + o(\epsilon^0)} x \sim (B_1(k) + o(\epsilon^0)) x, \quad x \rightarrow 0, \quad (3.6)$$

where $B_1(k) = \frac{A_2(k)}{A_4(k)}$.

This signifies that the derivative $\hat{u}'_x(t, 0)$ tends to a finite limit as $\epsilon \rightarrow 0-$.

If $-\frac{1}{2} > k > -1$, then the leading term of the denominator is $A_4(k) \epsilon^0$. Otherwise, if $k > -\frac{1}{2}$, then this leading term is $A_3(k) \epsilon^{-2k-1}$. Thus we have for $-\frac{1}{2} > k > -1$

$$\hat{u}(t, x) \sim \frac{A_1(k) \epsilon^{-2k-2} + o(\epsilon^{-2k-2})}{A_4(k) \epsilon^0 + o(\epsilon^0)}$$

and

$$\hat{u}'_x(t, 0) \sim B_2(k) \cdot \frac{1}{\epsilon^{2k+2}}, \quad B_2(k) = \frac{A_1}{A_4}, \quad x \rightarrow 0, \epsilon \rightarrow 0-. \quad (3.7)$$

At last for $k > -\frac{1}{2}$ we have

$$\hat{u}(t, x) \sim \frac{A_1(k)\epsilon^{-2k-2} + o(\epsilon^{-2k-2})}{A_3(k)\epsilon^{-2k-1} + o(\epsilon^{-2k-1})} x, \quad x \rightarrow 0, \epsilon \rightarrow 0-,$$

and

$$\hat{u}'_x(t, 0) \sim B_3(k) \cdot \epsilon^{-1}, \quad B_3(k) = \frac{A_1(k)}{A_3(k)} = 2k + 1. \quad (3.8)$$

If $k \in \mathbb{Z}$, then the numerator and the denominator in the leading term in the expansion of (3.5) as $x \rightarrow 0$ are expressed either through rational functions ($k \geq 0$) or through a Gaussian distribution function ($k < 0$). For $k = \frac{2l+1}{2}$, $l \in \mathbb{Z}$, the coefficient of the leading term is expressed through a fraction of series consisting of the digamma functions. Anyway, the asymptotics (3.6) takes place also for $k = \frac{l}{2}$, $l \in \mathbb{Z}$, $k \neq -\frac{1}{2}$, and it can be found also as a limit $\kappa \rightarrow k$. For $k < -1$ the function $\hat{u}(t, x)$ behaves as in (3.6), where the coefficient $B_1(k)$ can be calculated either independently or as $\lim_{\kappa \rightarrow k} \frac{A_2(\kappa)}{A_4(\kappa)}$. Since for $k = -1$ the degrees in ϵ^{-2k-2} and ϵ^0 coincide, then

$$\hat{u}(t, x) \sim \lim_{\kappa \rightarrow -1} \frac{(A_1(\kappa) + A_2(\kappa))\epsilon^0 + o(\epsilon^0)}{A_4(\kappa)\epsilon^0 + o(\epsilon^0)} x \sim (B_4 + o(\epsilon^0)) x, \quad x \rightarrow 0, \epsilon \rightarrow 0-,$$

where $B_4 = \lim_{\kappa \rightarrow -1} \left(B_1(\kappa) + \frac{A_1(\kappa)}{A_4(\kappa)} \right) = \frac{3|\alpha|}{2} - \frac{\sqrt{6}|\alpha|^{\frac{5}{2}}}{\sigma\sqrt{\pi}}$. For $k \geq 0$ the function $\hat{u}(t, x)$ has the asymptotics (3.8) with the same value $B_3(k)$. An exceptional case is $k = -\frac{1}{2}$, where

$$F_1(\epsilon, -1/2, \alpha, \sigma) = \bar{A}_1 \epsilon^{-1} + o(\epsilon^{-1}), \quad \bar{A}_1 = \lim_{\kappa \rightarrow -1/2} A_1 = \frac{4\sqrt{6}|\pi\alpha|^{\frac{3}{2}}}{\sigma},$$

$$F_2(\epsilon, -1/2, \alpha, \sigma) = A_5 \ln(-\epsilon) + o(\ln(-\epsilon)), \quad A_5 = -\bar{A}_1, \quad \epsilon \rightarrow 0-.$$

Thus, for $k = -\frac{1}{2}$ we have

$$\hat{u}_x(t, 0) \sim -\frac{1}{\epsilon \ln(-\epsilon)} + o\left(\frac{1}{\epsilon \ln(-\epsilon)}\right), \quad \epsilon \rightarrow 0-. \quad (3.9)$$

The following theorem summarizes our results:

Theorem 3.2. *Assume that in the case $n = 1$ the initial distribution function is $f(x) = \text{const} \cdot (1 + |x|^2)^k$, $k \in \mathbb{R}$, and the initial velocity has the form $u_0(x) = \alpha x$, $\alpha < 0$. Then the space derivative of the mean $\hat{u}(t, x)$ at the origin $x = 0$ is bounded for all $t > 0$ except for the critical time $T = -\frac{1}{\alpha}$. At the critical time the behavior of the derivative depends on k . Namely, for $k > -1$ the mean $\hat{u}(t, x)$ keeps the property of solutions to the non-perturbed Burgers equation to blow up at the critical time T at $x = 0$. The rates of the blowup for $-\frac{1}{2} > k > -1$, $k > -\frac{1}{2}$ and $k = -\frac{1}{2}$ are indicated in (3.7), (3.8) and (3.9), respectively. Otherwise, if $k \leq -1$, the derivative $\hat{u}'_x(t, 0)$ at the critical time remains bounded, i.e the singularity disappears.*

4. PRESSURELESS GAS DYNAMICS MODEL AND A LIMIT CASE AT VANISHING NOISE

Let us consider the pair $(\rho_t, u(\cdot, t)), t \geq 0$, where ρ_t is the probability distribution of the random variable X_t , governed by the SDE $X_t = X_0 + \int_0^t \mathbb{E}[u_0(X_0)|X_s] ds$, $t \geq 0$, with a given random variable X_0 with values in \mathbb{R} and function $u_0(x)$, and $u(t, x) = \mathbb{E}[u_0(X_0)|X_t = x]$. According to [7], $(\rho_t, u(\cdot, t), t \geq 0)$ is a weak solution to the pressureless gas dynamics model

$$\partial_t \rho + \partial_x(\rho u) = 0, \quad \partial_t(\rho u) + \partial_x(\rho u^2) = 0. \quad (5.1)$$

Therefore it is natural to expect that the limit as $\sigma \rightarrow 0$ of the mean \hat{u} (denoted by $v(t, x)$) takes part in the solution to (5.1).

For smooth $u_0(x)$ and $f(x)$ this can be readily shown. First of all, let us introduce the function $\rho(t, x) = \int_{\mathbb{R}^n} P(t, x, u) du$ and notice that it satisfies the continuity equation $\partial_t \rho + \partial_x(\rho \hat{u}) = 0$. Further, we check that the function $v(t, x)$ satisfies the Burgers equation (1.1). Indeed, $\frac{t^{3/2}}{\sigma\sqrt{6\pi}} \exp\left(-\frac{3(u_0(s)t + (s-x))^2}{2\sigma^2 t^3}\right) \rightarrow \delta(s - s(t, x))$, as $\sigma \rightarrow 0$ in \mathcal{D}' , where $s(t, x)$ is a solution to equation $u_0(s) + \frac{s-x}{t} = 0$, given in implicit form. This function exists and it is differentiable provided $t \neq -\frac{1}{u'_0(x)}$ (at this moment of time the solution to the Burgers equation blows up). Thus, $\hat{u}(t, x) \rightarrow u_0(s(t, x))$ as $\sigma \rightarrow 0$. Now it is sufficient to substitute $u_0(s(t, x))$ into (1.1) and compute the derivatives of $s(t, x)$ by means of the implicit function theorem.

Thus, for smooth initial data $(f(x), u_0(x))$ the pair (ρ, \hat{u}) is a solution to the system

$$\partial_t \rho + \partial_x(\rho \hat{u}) = 0, \quad \partial_t(\rho \hat{u}) + \partial_x(\rho \hat{u}^2) = \Lambda, \quad \Lambda = - \int_{\mathbb{R}} P_x(t, x, u) (u - \hat{u})^2 du,$$

where $\Lambda \rightarrow 0$ as $\sigma \rightarrow 0$. The integral relaxation term Λ can be used instead of the traditional viscosity [8].

It is interesting to consider the Fokker-Planck equation (1.8) as a kinetic equation

$$\frac{\partial P(t, x, u)}{\partial t} + \sum_{k=1}^n \left(u_k \frac{\partial P(t, x, u)}{\partial x_k} + \frac{\partial \dot{u}_k P(t, x, u)}{\partial u_k} \right) = 0$$

(e.g.[9]), where the acceleration \dot{u} of the particles is due to external forces and the interaction forces with other particles. It can be readily calculated that in our case $\dot{u} = \frac{2}{t}(\hat{u} - u)$.

The authors would like to thank an anonymous referee for helpful suggestions.

REFERENCES

- [1] H. Risken, *The Fokker-Planck Equation Methods of Solution and Applications*, Second Edition, Springer-Verlag, 1989.
- [2] Weinan E, K. M. Khanin, A. E. Mazel, Ya. G. Sinai, *Invariant measures for Burgers equation with stochastic forcing*, Ann. of Math. (2) 151, no. 3, 877-960 (2000).
- [3] J.P.Bouchaud, M.Mézard, *Velocity fluctuations in forced Burgers turbulence*, Phys.Rev. E**54**, 5116 (1996).
- [4] V.Gurarie, *Burgers equations revisited*, arXiv:nlin/0307033v1 [nlin.CD] (2003).
- [5] S.Albeverio, O.Rozanova, *The non-viscous Burgers equation associated with random positions in coordinate space: a threshold for blow up behaviour*, to appear in M³AS (Mathematical methods and models in applied science)(2009), arXiv:0708.2320v2 [math.AP].

- [6] I.S. Gradshteyn, I.M. Ryzhik, Table of integrals, series, and products. 6th ed. San Diego, CA: Academic Press, 2000.
- [7] A.Dermoune, *Probabilistic interpretation for system of conservation law arising in advection particle dynamics* C.R.Acad.Sci.Paris, **326**, Serie I, 595-599 (1998)
- [8] D.Tan, T.Zhang, Y.Zheng, *Delta-shock waves as limits of vanishing viscosity for hyperbolic system of conservation laws*, J.Diff.Equat., **112** (1994), 1-32.
- [9] H.Struchtrup, *Macroscopic Transport Equations for Rarefied Gas Flows: Approximation Methods in Kinetic Theory*, Springer-Verlag Berlin Heidelberg, 2005.

(¹) UNIVERSITÄT BONN, INSTITUT FÜR ANGEWANDTE MATHEMATIK, ABTEILUNG FÜR STOCHASTIK, WEGELERSTRASSE 6, D-53115, BONN; HCM, SFB611 AND IZKS, BONN; BiBoS, BIELEFELD–BONN, GERMANY

(²) MATHEMATICS AND MECHANICS FACULTY, MOSCOW STATE UNIVERSITY, MOSCOW 119992, RUSSIA

E-mail address, ¹: `albeverio@uni-bonn.de`

E-mail address, ²: `rozanova@mech.math.msu.su`